Note on the improved Gibbs weighted-mean algorithm

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SUMMARY

An improved Gibbs weighted-mean algorithm, which is an algorithm for determining a hypothtical composite clock from an ensemble of clocks, was proposed in [2]. A part of the algorithm is modified and the underlying philosophy of the procedure is elucidated by using a maximal-likelihood-sense criterion.

1. A clock model

It is known [1], [4] that, as a general model of clock behavior, the measured time departure of a clock h from an ideal clock at date t is expressed by

$$(1.1)\quad x(t\,;h)=a(h)+b(h)t+(1/2)c(h)t^2+\varepsilon(t\,;h),$$

where a(h) is the time offset, b(h) is the frequency offset (or the time drift), and c(h) is the frequency drift. All of these quantities are assumed to be constants. The random time fluctuations, typified by $\varepsilon(t;h)$, are generated by the random walk rule,

$$(1.2)\quad \begin{aligned}\varepsilon(t\,;h)&=\varepsilon(t-1\,;h)+\zeta(t\,;h), && [t=1,2,3,\ldots]\\ &=\zeta(0\,;h), && [t=0]\end{aligned}$$

where the sequence of random variables $\{\zeta(t;h):\ t=0,1,2,\ldots\}$ is independent and identically distributed. For cesium clocks, $\zeta(t;h)$'s are known to be normally distributed. So we assume that $\zeta(t;h)\sim N(0, \mathrm{Var}\{\zeta(0;h)\})$. Since we know by (1.2) that

$$\varepsilon(t\,;h)=\sum_{s=0}^{t}\zeta(s\,;h),$$

the retrospective first difference of the clock h is written as

$$(1.3)\quad \begin{aligned}y(t\,;h)&=x(t-1\,;h)\\ &=b(h)+(1/2)c(h)(2t-1)+\varepsilon(t\,;h)-\varepsilon(t-1\,;h)\\ &=b(h)+(1/2)c(h)(2t-1)+\zeta(t\,;h).\end{aligned}$$

So the retrospective second difference of the clock h is given by

$$(1.4)\quad \begin{aligned} z(t\,;h) &= y(t\,;h)-y(t-1\,;h)\\ &= c(h)+\varepsilon(t\,;h)-2\varepsilon(t-1\,;h)+\varepsilon(t-2\,;h)\\ &= c(h)+\zeta(t\,;h)-\zeta(t-1\,;h). \end{aligned}$$

Hence

$$(1.5)\quad z(t\,;h)\sim N(c(h),\ 2\mathrm{Var}\{\zeta(t\,;h)\}).$$

Now we consider a set H of m clocks. If the sequences $\{\zeta(t;h)\}$ $[h\epsilon H]$ are mutually independent, then the joint distribution function of the random variables

$$(1.6)\quad z(t\,;h)-z(t-1\,;h)=\zeta(t\,;h)-2\zeta(t-1\,;h)+\zeta(t-2\,;h)\quad [h\,\epsilon\, H]$$

is written as

$$(1.7)\quad L=\prod_{h\in H}(1/\sqrt{2\pi\sigma(h)})\exp\{-(z(t\,;h)-z(t-1\,;h))^2/2\sigma^2(h)\}$$

where

$$(1.8)\quad \sigma^2(h)=6\mathrm{Var}\{\zeta(0\,;h)\}\quad [h\,\epsilon\, H].$$

We set up a problem: Given $z(t-1;h)$ and $\sigma^2(h)$ $[h\epsilon H]$, estimate $z(t;h)$ $[h\epsilon H]$ in such a way that (1.7) takes a maximum.

The values $z(t;h)$ $[h\epsilon H]$, which attain a maximum, are considered to be the most probable outcomes of the corresponding random variables.

In order to solve the problem we take the logarithm of both side of (1.7):

$$(1.9)\quad 2\log L=-m\log 2\pi-\sum_{h\in H}\log\sigma^2(h)-\sum_{h\in H}(z(t\,;h)-z(t-1\,;h))^2/\sigma^2(h).$$

Hence to maximize (1.7) with respect to $z(t;h)$ $[h\epsilon H]$ is to minimize the last term of (1.9):

$$(1.10)\quad \sum_{h\in H}(z(t\,;h)-z(t-1\,;h))^2/\sigma^2(h).$$

Next we consider another problem: Given $z(t-1;h)$ and $z(t;h)$ $[h\epsilon H]$, estimate $\sigma^2(h)$ $[h\epsilon H]$ in such a way that (1.7) takes a maximum.

Differentiating (1.9) with respect to $\sigma^2(h)$ and setting it to zero we have

$$(1.11)\quad \sigma^2(h)=(z(t;h)-z(t-1;h))^2 \quad [h\in H].$$

These are estimates of the variances $\sigma^2(h)$ $[h\in H]$.

2. A weighted-mean least-squares algorithm

In [2] Endow proposed a weighted-mean least-squares algorithm for the construction of a candidate as an LCC (local composit clock). The weight $w(t;h,H)$ in the algorithm does not seem to express the relationship with the deviation (or the variance) of the random time fluctuations of the clock h. Here we therefore intend to improve the algorithm, and to elucidate the underlying philosophy of the procedure.

Most of the definitions and notations that we shall use in the sequel follow those of NPL DES Memorandum [3].

In consideration of the frequency drift of clocks the algorithm that we shall expound handles explicitly not the SID data, typified by

$$(2.1)\quad x(t;h,h')=x(t;h)-x(t;h'),$$

but the retrospective second differences (with respect to t), typified by

$$(2.2)\quad \begin{aligned} z(t;h,h') &= x(t;h,\ h)-2x(t-1;h,h')+x(t-2;h,h')\\ &= y(t;h,h')-y(t-1;h,h'), \end{aligned}$$

where $y(t;h,h')$ is the retrospective first difference (with respect to t):

$$y(t;h,h')=x(t;h,h')-x(t-1;h,h').$$

Since it is immediate that

$$z(t;h,h')=z(t;h)-z(t;h'),$$

the quantity $z(t;h,h')$ may be regarded, by (1.4), as equal to the average mean-daily drift plus random fluctuations of the clock h with respect to the clock h′ and is, by definition, antisymmetric in the arguments h and h′:

$$(2.3)\quad z(t;h,h')=-z(t;h',h).$$

We shall estimate the quantity $z(t;h)$ by the retrospective second difference with respect to the composite clock c, typified by $z(t;h,c)$.

Let us suppose that the retrospective second difference of the clock h with respect to the composit clock c is equal to the retrospective second difference of the clock h with respect to the ideal clock:

$$(2.4)\quad z(t\,;h,c)=z(t\,;h)\quad [h\in H],$$

and

$$(2.5)\quad \sigma^2(t\,;h,c)=\sigma^2(h)\quad [h\in H].$$

Then the joint distribution function of the difference $z(t;h,c)-z(t-1;h,c)$ $[h\in H]$ is given, by (1.7), as

$$(2.6)\quad L=\prod_{h\in H}(1/\sqrt{2\pi\sigma(t\,;h,c)})\times\exp\{-z(t\,;h,c)-z(t-1\,;h,c))^2/2\sigma^2(t\,;c)\}.$$

So we can estimate the quantities $z(t;h,c)$ and $\sigma^2(t;h,c)$ $[h\in H]$, in the same way as in 1, by maximizing (2.6) in terms of the previous quantities $z(t-1;h,c)$ and $\sigma^2(t-1;h,c)$ $[h\in H]$.

The assumptions (2.4) and 2.5) state that the composite clock c should be equal to the ideal clock.

Hence we set up the following problem: (a) Given the values $z(t;h,k)$, $z(t-1;h,c)$ and $\sigma^2(t-1;h,c)$ $[h,k\in H]$, estimate $z(t;h,c)$ $[h\in H]$ in such a way that

$$(2.7)\quad \sum_{h\in H}(1/\sigma^2(t-1\,;h,c))(z(t\,;h,c)-z(t-1\,;h,c))^2\quad [h\in H]$$

takes a minimum value under the constraints

$$(2.8)\quad z(t\,;h,c)-z(t\,;k,c)=z(t\,;h,k)\quad [h,\ k\in H].$$

(b) Using the result in (a) determine the values

$$(2.9)\quad \sigma^2(t\,;h,c)=(z(t\,;h,c)-z(t-1\,;h,c))^2\quad [h\in H].$$

Putting

$$(2.10)\quad w(t-1\,;h,c)=(1/\sigma^2(t-1\,;h,c))/\sum_{k\in H}(1/\sigma^2(t-1\,;k,c)),$$

and introducing the Lagrange multipliers, we can reduce the minimization problem

(a) to the folowing unconstraints problem:

Minimize $Q(z(t\,;h(1),c),\dots,z(t\,;h(m),c);L(t\,;1),\dots,L(t\,;m-1))$,

where

$$(2.11)\quad Q=\sum_{i=1}^{m} w(t-1\,;h(i),c)(z(t\,;h(i),c)-z(t-1\,;h(i),c))^2$$
$$+2\sum_{i=1}^{m-1} L(t\,;i)[z(t\,;h(i),c)-z(t\,;h(i+1),c)-z(t\,;h(i),h(i+1)].$$

A necessary condition that Q takes a minimum is the following system of equalities:

$$\partial Q/\partial z(t\,;h(i),c)=0 \quad [i=1,\dots,m],$$
$$\partial Q/\partial L(t\,;i)=0 \qquad [i=1,\dots,m-1].$$

Performing the differentiations indicated, we have

$$(2.12)\quad w(t-1\,;h(i),c)(z(t\,;h(i),c)-z(t-1\,;h(i),c))$$
$$-\varDelta(i,1)L(t\,;i-1)+\varDelta(i,m)L(t,i)=0 \quad [i=1,\dots,m-1]$$

where $\varDelta(i,j)=0$ if $i=j$; $=1$ if $i\neq j$. The simultaneous equations (2.12) and (2.13) will be written in a matrix form

$$(2.14)\quad Az=b,$$

where

$$A=\begin{pmatrix} B & C \\ C' & 0 \end{pmatrix}$$

$$B=[\delta(i,j)w(t-1\,;h(i),H)]:(m\times m),$$

$$C=\begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & 0 & \\ & -1 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & -1 & 1 \\ & & & & & -1 \end{bmatrix} : m\times(m-1),$$

$$z = \begin{pmatrix} z_1 \\ L \end{pmatrix} : (2m-1)\times 1,$$

$$z_1 = [z(t\,;h(i), c)] : m\times 1,$$

$$L = [L(t\,;i)] : (m-1)\times 1,$$

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} : (2m-1)\times 1,$$

$$b_1 = [w(t-1\,;h)i, c)] : m\times 1,$$

$$b_2 = [z(t\,;h(i), h(i+1)] : (m-1)\times 1,$$

and $\delta(i, j)$ is the Kroneker delta defined by

$$\delta(i, j) = 1 \text{ if } i=j\,; =0 \text{ if } i \neq j.$$

Since we know that the coefficient matrix A is non-singular (cf. Appendix 1 of [2]), Equation (2.14) has a unique solution z. Thus we have determined the quantities $z(t; h, c)$ $[h \epsilon H]$.

The determination of $\sigma^2(t; h, c)$ $[h \epsilon H]$ is straightforward by (2.9).

ALGORITHM

(1) Quantify the initial set of weights $\{w(0; h, c) : h \epsilon H\}$ by

$$w(0; h, c) = 1/m \quad [h \epsilon H],$$

where m denotes the number of clocks in the est T.

(2) Determine the initial set $\{z(0; h, c) : h \epsilon H\}$ by

$$z(0; k, v) = \sum_{H(K)} w(0; h, c) z(0; k, h) \quad [h \epsilon H],$$

where $H(k) = H - \{k\}$.

(3) Solve Equation (2.14) for z, using the immediately preceding quantities $\{z(t-1; h, c) : h \epsilon H\}$ and $\{w(t-1; h, c) : h \epsilon H\}$ and current SID data $\{z(t; h, h') : h, h \epsilon H\}$.

(4) Calculate the variances $\{\sigma^2(t; h, c) : h \epsilon H\}$ and the weights $\{w(t; h, c) : h \epsilon H\}$ by Equations 2.9) and (2.10) respectively.

(5) Return to (3), upon getting the SID data $\{x(t; h, h') : h, h' \epsilon H\}$.

REFERENCES

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